Math 10A
Midterm II; Thursday, 7/19/2018
Time: 2:10 PM **Name:** ____________________
Instructor: Roy Zhao **Student ID:** ____________________

- **DO NOT OPEN THE MIDTERM UNTIL TOLD TO DO SO!**
- Do all problems as best as you can. The exam is 80 minutes long. You may not leave during the last 30 minutes of the exam.
- Use the provided sheets to write your solutions. You may use the back of each page for the remainder of your solutions; in such a case, put an arrow at the bottom of the page and indicate that the solution continues on the back page. **No extra sheets of paper can be submitted with this exam!**
- The exam is closed notes and book, which means: **no class notes, no review notes, no textbooks, and no other materials can be used during the exam.** You can only use your cheat sheet. The cheat sheet is one side of one regular 8×11 sheet, handwritten.
- **NO CALCULATORS ARE ALLOWED DURING THE EXAM!**
- Justify all your answers, include all intermediate steps and calculations, and box your answers.

1. (22 points) Calculate the following integrals and derivatives.

   (a) (4 points) $\int e^{2x} dx =$

   Solution: $\dfrac{e^{2x}}{2} + C.$

   (b) (5 points) $\displaystyle\int_{-5}^{5} \frac{\sin(x)}{x^4 + 3x^2 + 1} =$

   Solution: 0 because the function is odd.

   (c) (6 points) $\displaystyle\int_0^{\sqrt{\pi/2}} x\cos(x^2)dx =$

   Solution: We use u substitution and set $u = x^2$ so $du = 2xdx$ or $dx = \frac{du}{2x}$ so

$$\int_0^{\sqrt{\pi/2}} x\cos(x^2)dx = \int_0^{\pi/2} \frac{\cos(u)}{2} du = \frac{\sin(u)}{2}\bigg|_0^{\pi/2} = \frac{\sin(\pi/2)}{2} - \frac{\sin(0)}{2} = \frac{1}{2}.$$

   (d) (7 points) $\displaystyle\frac{d}{dx}\int_x^{x^3} \frac{t\sin(t)}{e^t} dt =$

   Solution: Using FTC, we get

$$\frac{x^3\sin(x^3)}{e^{x^3}} \cdot 3x^2 - \frac{x\sin(x)}{e^x} \cdot 1$$

2. (16 points) (a) (12 points) Calculate $\int \frac{x^2+1}{x^2-1}dx$.

Solution: First we need to long divide to get $\frac{x^2+1}{x^2-1} = 1 + \frac{2}{x^2-1}$. We factor $x^2-1 = (x-1)(x+1)$. Then we write $\frac{2}{x^2-1} = \frac{A}{x-1} + \frac{B}{x+1}$. Multiplying gives $2 = A(x+1) + B(x-1)$. Plug in $x = 1$ to get $2 = 2A$ so $A = 1$ and plug in $x = -1$ to get $2 = -2B$ so $B = -1$ and hence

$$\int \frac{x^2+1}{x^2-1}dx = \int 1 + \frac{1}{x-1} - \frac{1}{x+1}dx = x + \ln|x-1| - \ln|x+1| + C.$$

(b) (4 points) Set up the partial fractions decomposition of $\frac{3x^2+2x-4}{(x+1)(x^2-1)(x^2+1)^3}$.
(you do not need to solve for the constants)

Solution: Simplify the denominator as $(x+1)(x+1)(x-1)(x^2+1)^3 = (x-1)(x+1)^2(x^2+1)^3$.

$$\frac{A}{x-1} + \frac{B}{x+1} + \frac{C}{(x+1)^2} + \frac{Dx+E}{x^2+1} + \frac{Fx+G}{(x^2+1)^2} + \frac{Hx+I}{(x^2+1)^3}.$$

3. (16 points) Integrate $\int e^x \cos(2x)dx$.

Solution: Let $u = \cos(2x)$ and $dv = e^x dx$. Then $du = -2\sin(2x)dx, v = e^x$ so

$$\int e^x \cos(2x)dx = e^x \cos(2x) - \int -2e^x \sin(2x)dx = e^x \cos(2x) + 2\int e^x \sin(2x)dx.$$

We use integration by parts again with $u = \sin(2x), dv = e^x dx$ so $du = 2\cos(2x)dx, v = e^x$. So we get

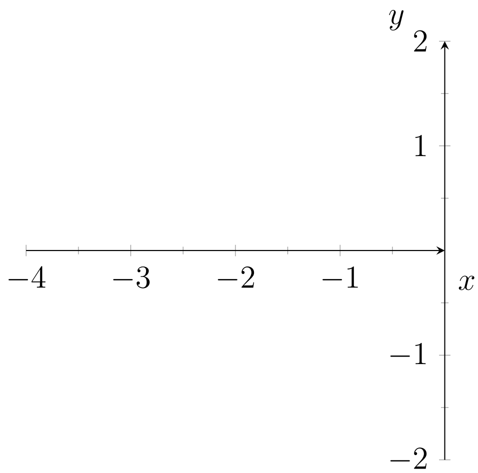
$$\int e^x \cos(2x)dx = e^x \cos(2x) + 2e^x \sin(2x) - 4\int e^x \cos(2x)dx.$$

Adding the integral to the left side, we get

$$\int e^x \cos(2x)dx = \frac{e^x \cos(2x) + 2e^x \sin(2x)}{5} + C.$$

Answer: $\int e^x \cos(2x)dx =$

4. (22 points) (a) (10 points) Use the Trapezoid method with $n = 2$ to integrate $\int_{-3}^{-1} \frac{1}{x} dx$. Sketch the function as well as what area your approximation calculates.



Solution: Our $\Delta x = \frac{-1-(-3)}{2} = 1$ and so our intervals are $[-3,-2],[-2,-1]$. Now the trapezoid rule gives us the area as

$$\frac{\Delta x}{2}(f(-3) + 2f(-2) + f(-1)) = \frac{1}{2}\left[\frac{1}{-3} + \frac{2}{-2} + \frac{1}{-1}\right] = \frac{1}{2} \cdot \frac{-7}{3} = \frac{-7}{6}.$$

(b) (4 points) Without calculating the integral, is this an overestimate or underestimate?

Solution: Since we are using the trapezoid rule, we look at the second derivative. The second derivative is $\frac{2}{x^3}$ and for $x \in [-3,-1]$, x is negative so $\frac{2}{x^3}$ is negative. Since the second derivative is negative, the trapezoid rule gives us a $\boxed{\textit{underestimate}}$.

(c) (8 points) Without calculating the integral, is this approximation within 0.5 of the actual answer?

Solution: The bound of our error is

$$E_T = \frac{K_2(b-a)^3}{12n^2}.$$

We have $b = -1, a = -3, n = 2$. So we need to calculate $K_2 = \max|2/x^3|$. The critical points are when $(2/x^3)' = -6/x^3 = 0$ which is never. Thus, we only need to plug in the endpoints which are $-1, -3$ to get $|2/(-1)^3| = 2$ and

$|2/(-3)^3| = 2/27 < 2$. So $K_2 = 2$. Therefore, our error is less than

$$E_T = \frac{2(-1-(-3))^3}{12 \cdot 2^2} = \frac{2 \cdot 8}{12 \cdot 4} = \frac{1}{3} < 0.5.$$

So, we are within 0.5 of the actual answer.

5. (16 points) (a) (8 points) Calculate $\int_e^\infty \frac{1}{x(\ln x)^2} dx$.

Solution: We write

$$\int_e^\infty \frac{1}{x(\ln x)^2} dx = \lim_{n\to\infty} \int_e^n \frac{1}{x(\ln x)^2} dx.$$

Then we u sub $u = \ln x$ so $du = \frac{1}{x} dx$ and $\ln e = 1$

$$= \lim_{n\to\infty} \int_1^{\ln n} \frac{1}{u^2} du = \lim_{n\to\infty} \frac{-1}{u}\Big|_1^{\ln n} = \lim_{n\to\infty} \frac{-1}{\ln n} + \frac{1}{1} = 1.$$

(b) (8 points) Does $\int_e^\infty \frac{\cos^2(x)}{(x\ln x)^2 + e^{-x^2}} dx$ converge?

Solution: We have that $(x\ln x)^2 + e^{-x^2} \geq (x\ln x)^2 = x^2(\ln x)^2 \geq x(\ln x)^2$ so

$$\frac{\cos^2(x)}{(x\ln x)^2 + e^{-x^2}} \leq \frac{\cos^2(x)}{x(\ln x)^2} \leq \frac{1}{x(\ln x)^2}.$$

Since the integral of the right function converges (from the previous part), this integral converges.

6. (8 points) Bubble True or False. (1 point for correct answer, 0 if incorrect)

(a) (T) ● We can only split an integral along its interval as in $\int_a^b f(x)\,dx = \int_a^c f(x)\,dx + \int_c^b f(x)\,dx$ only when c is between a and b.

Solution: This is always true.

(b) ● (F) $\int_0^1 f'(x)\,dx = f(1) - f(0)$.

Solution: This is by FTC and the fact that f is an antiderivative of $f'(x)$.

(c) ● (F) Suppose that $f'''(x) = 5$ for all $x \in [a, b]$. Then, Simpson's rule computes $\int_a^b f(x)\,dx$ exactly.

Solution: If $f'''(x) = 5$, then $f^{(4)}(x) = 0$ and hence $K_4 = \max|f^{(4)}(x)| = \max|0| = 0$ so Simpson's rule has no error.

(d) ● (F) Assume that $f(x) \geq 0$. In order to show that the integral $\int_1^\infty \frac{1}{f(x)}dx$ converges, it suffices to find a function $g(x)$ such that $f(x) \geq g(x) \geq 0$ on $[1, \infty)$ and show that $\int_1^\infty \frac{1}{g(x)}dx$ converges.

Solution: This is true because if $f(x) \geq g(x)$, then $\frac{1}{f(x)} \leq \frac{1}{g(x)}$.

(e) (T) ● $\int_{-1}^2 \frac{dx}{x} = \ln|x| |_{-1}^2 = \ln 2 - \ln 1$.

Solution: We can't integrate over a gap which is the vertical asymptote at $x = 0$.

(f) (T) ● $\frac{d}{dx}\int_0^5 \sqrt{1-t}dt = \sqrt{1-x}$.

Solution: The derivative is just 0.

(g) ● Ⓕ If $f'(x) \leq g'(x) \leq 0$ for all $x \in [a, b]$, the error bound for using the left endpoint method to calculate $\int_a^b f(x)dx$ will be larger than for $\int_a^b g(x)dx$.

Solution: If $f'(x) \leq g'(x) \leq 0$, then $|f'(x)| \geq |g'(x)| \geq 0$ and hence K_1 will be larger for f than it will for g. So, the error bound will be larger for f than it will for g.

(h) Ⓣ ● The midpoint method will overestimate the integral $\int_0^1 x^3 dx$.

Solution: The second derivative is $6x \geq 0$ when $x \in [0, 1]$. So, the midpoint method will underestimate the area.